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Translated by L. K.

UDC 532.5

STRESS TENSOR AND AVERAGING IN MECHANICS OF CONTINUOUS MEDIA

PMM Vol. 39, №2, 1975, pp. 374-379
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(Received January 7, 1974)

The problem of determining the mean stress and the other macrovariables originates upon passing from the equations of motion which are valid in the microscale, to the macroscopic equations which describe the motion of continuous media (such as a turbulized fluid, an elastic medium with microdefects, the suspension of gas bubbles or solid particles in a fluid, etc.). The mean value of the stress tensor over a volume was introduced in the monograph [1], and precisely this quantity was used in the governing relations to compute the Einstein viscosity of suspensions. Moreover, some effective representation in terms of integrals over surfaces [1] was used in specific calculations of these means with respect to the volume. Later, Batchelor [2], and some other authors after him [3], used precisely these means with respect to the volume as the stresses in the macroequations of motion by assuming the equivalence between the average with respect to a volume and with respect to a surface. Hence, in particular, the absolute symmetry of the macrostress tensor follows in the above-mentioned cases.

In this paper it is shown that the average of the microstress tensor and the microflux of the momenta with respect to the volume according to the rule in [1] determines only some symmetric part of the complete macrostress tensor. For the simple case of a viscous fluid moving inhomogeneously over a microlevel, this mean of the tensor with respect to the volume is related linearly to the mean strain rates. Moreover, the representation used in [1] permits clarification of the essential difference between the mean stresses with respect to the volume and with respect to the surfaces, in the general case.

The method of integrating the microequations with respect to the vloume [4 – 6] naturally results in the appearance of stresses in the macroequations, which are the means with respect to the differential macroareas. It is essential that the macrostress tensory is hence generally nonsymmetric although the equations of motion in the microscale correspond to symmetric continuum mechanics. It is this consideration which permitted the development of the continuum equations of motion of a suspension, which reflects the effect of nonequilibrium intrinsic rotation of the suspended particles [7], and the case of a turbulized fluid with anisotropies of eddy character is set in conformity to the nonzero antisymmetric part of the Reynolds stresses [8].

1. Let us separate the two scales of the investigation of complex flows, the microscale and the macroscale, and let us seek the continuum macroequations by considering that the continuum equations for the microscale are known. To wit, in each differential microelement $dV = dx_1 dx_2 dx_3$, let the following equations of mass and momentum balance be satisfied:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \tag{1.1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i \tag{1.2}$$

where ρ is the density, u_i is the local velocity, σ_{ij} is the microstress tensor, F_i is the mass force. In case the microtensor σ_{ij} is nonsymmetric, (1.1) and (1.2) must be supplemented by the balance equation for the moment of momentum. However, let us here limit ourselves to the consideration of the case of a symmetric microstress, related linearly to the strain rate tensor (A_{ijkl} is the tensor of viscosity coefficients)

$$\sigma_{ij} \equiv \sigma_{ji} = {}^1/_2 A_{ijkl} (\partial u_k / \partial x_l + \partial u_l / \partial x_k) - p\delta_{ij} \tag{1.3}$$

Multiplying (1.2) by the coordinate x_k results in the relationship

$$\sigma_{ik} - \rho u_i u_k = -\frac{\partial \rho u_i x_k}{\partial t} - \frac{\partial \rho u_i u_j x_k}{\partial x_j} + \frac{\partial \sigma_{ij} x_k}{\partial x_j} + F_i x_k \tag{1.4}$$

which permits [1, 2] direct expression of the momentum flux tensor. If (1.4) is multiplied by the alternating Levi-Civitta tensor ε_{lik}, then it goes over into the balance equation of the moment of momentum

$$\frac{\partial \varepsilon_{lik} \rho u_i x_k}{\partial t} + \frac{\partial \varepsilon_{lik} \rho u_i x_k u_j}{\partial x_j} = \frac{\partial \varepsilon_{lik} \sigma_{ij} x_k}{\partial x_j} + \varepsilon_{lik} F_i x_k \tag{1.5}$$

Here the symmetry condition of the momentum microflux $\varepsilon_{lik} (\sigma_{ik} - \rho u_i u_k) = 0$ has been taken into account.

Integrating (1.1) − (1.5) with respect to the volume V, the Ostrogradskii-Gauss theorem can be applied in the case of continuous fields of variables

$$\frac{\partial}{\partial t} \int_V \rho\, dV + \int_S \rho u_j dS_j = 0 \tag{1.6}$$

$$\frac{\partial}{\partial t} \int_V \rho u_i\, dV + \int_S \rho u_i u_j dS_j = \int_S \sigma_{ij} dS_j + \int_V F_i\, dV$$

$$\int_V \sigma_{ij}\, dV = \frac{1}{2} A_{ijkl} \int_V \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) dV - \int_V p\delta_{ij} dV$$

$$\int_V (\rho u_i u_j - \sigma_{ik})\, dV = \frac{\partial}{\partial t} \int_V \rho u_i x_k\, dV + \int_S \rho u_i u_j x_k dS_j - \int_S \sigma_{ij} x_k dS_j - \int_S F_i x_k dV$$

$$\frac{\partial}{\partial t} \int_V \varepsilon_{lik} \rho u_i x_k\, dV + \int_S \varepsilon_{lik} \rho u_i x_k u_j dS_j = \int_S \varepsilon_{lik} \sigma_{ij} x_k dS_j + \int_V \varepsilon_{lik} F_i x_k dV$$

where

$$\varepsilon_{lik} \int_V \sigma_{ik}\, dV = \varepsilon_{lik} \int_V \rho u_i u_k\, dV = 0$$

and the surface integral is taken over the whole surface S of the volume V. In the absence of inertial and mass forces, the fourth relationship in (1.6) agrees with the Landau and Lifshits representation [1], connecting the mean of the stress tensor with repect to the volume and the integral of the surface forces.

2. Let V be the elementary macrovolume $V = \Delta X_1 \Delta X_2 \Delta X_3$, where X_i are the macrocoordinates of the center of mass of the volume V. Now, if Eqs. (1.5) are divided by the volume V, they go over into the averaged equations

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial \langle \rho u_j \rangle_j}{\partial X_j} = 0 \tag{2.1}$$

$$\frac{\partial \langle \rho u_i \rangle}{\partial t} + \frac{\partial \langle \rho u_i u_j \rangle_j}{\partial X_j} = \frac{\partial \langle \sigma_{ij} \rangle_j}{\partial X_j} + \langle F_i \rangle \tag{2.2}$$

$$\langle \sigma_{ij} \rangle = \frac{A_{ijkl}}{2} \left(\frac{\partial \langle u_k \rangle_l}{\partial X_l} + \frac{\partial \langle u_l \rangle_k}{\partial X_k} \right) - \langle p \delta_{ij} \rangle \tag{2.3}$$

$$\langle \rho u_i u_k \rangle - \langle \sigma_{ik} \rangle = \frac{\partial \langle \rho u_i x_k \rangle}{\partial t} + \frac{\partial \langle \rho u_i u_j x_k \rangle_j}{\partial X_j} - \frac{\partial \langle \sigma_{ij} x_k \rangle_j}{\partial X_j} - \langle F_i x_k \rangle \tag{2.4}$$

$$\frac{\partial \langle \varepsilon_{lik} \rho u_i x_k \rangle}{\partial t} + \frac{\partial \langle \varepsilon_{lik} \rho u_i x_k u_j \rangle_j}{\partial X_j} = \frac{\partial \langle \varepsilon_{lik} \sigma_{ij} x_k \rangle_j}{\partial X_j} + \langle \varepsilon_{lik} F_i x_k \rangle \tag{2.5}$$

where

$$\varepsilon_{lik} \langle \sigma_{ik} \rangle = \varepsilon_{lik} \langle \rho u_i u_k \rangle = 0$$

and the meaning of the averaging symbols is illustrated as follows:

$$\langle \sigma_{ij} \rangle = \frac{1}{V} \int_V \sigma_{ij} \, dV, \quad \langle \sigma_{ij} \rangle_j = \frac{\Delta X_j}{\Delta X_1 \Delta X_2 \Delta X_3} \int_S \sigma_{ij} n_j \, dS, \quad dS_j = n_j \, dS$$

where S_j is the area of the faces of the volume $V = \Delta X_1 \Delta X_2 \Delta X_3$ with the normal n_j.

If (2.2) is now multiplied by X_k, we then obtain

$$\langle \rho u_i u_k \rangle_k - \langle \sigma_{ik} \rangle_k = \frac{\partial \langle \rho u_i X_k \rangle}{\partial t} + \frac{\partial \langle \rho u_i u_j X_k \rangle_j}{\partial X_j} - \frac{\partial \langle \sigma_{ij} X_k \rangle_j}{\partial X_j} - \langle F_i X_k \rangle \tag{2.6}$$

Subtracting this result from (2.4) determines the difference

$$\langle \sigma_{ik} \rangle_k - \langle \rho u_i u_k \rangle_k = \langle \sigma_{ik} \rangle - \langle \rho u_i u_k \rangle + \frac{\partial \langle \rho u_i \xi_k \rangle}{\partial t} + \frac{\partial \langle \rho u_i u_j \xi_k \rangle_j}{\partial X_j} - \langle F_i \xi_k \rangle - \frac{\partial \langle \sigma_{ij} \xi_k \rangle_j}{\partial X_j} \tag{2.7}$$

between the mean values of the momentum flux over the volume and over the surface. Here $\xi_k = x_k - X_k$ is the coordinate relative to the center of mass of the volume V. If (2.7) is multiplied by the tensor ε_{lik}, then the equation of the inner moment of momentum in the volume V

$$\varepsilon_{lik} \langle \sigma_{ik} - \rho u_i u_k \rangle_k = \frac{\partial \langle \varepsilon_{lik} \rho u_i \xi_k \rangle}{\partial t} + \frac{\partial \langle \varepsilon_{lik} \rho u_i u_j \xi_k \rangle_j}{\partial X_j} - \frac{\partial \langle \varepsilon_{lik} \sigma_{ij} \xi_k \rangle_j}{\partial X_j} - \langle \varepsilon_{lik} F_i \xi_k \rangle \tag{2.8}$$

will be the result.

Therefore, the macrostress tensor $\langle \sigma_{ij} \rangle_j$, which is the mean with respect to the surface in conformity with the initial Cauchy representations [9] enters the averaged momentum equation. According to (2.7), the mean of the stress tensor with respect to the volume $\langle\sigma_{ij}\rangle$ is generally only a part of the mean with respect to the surface $\langle\sigma_{ij}\rangle_j$ and cannot be used directly in the momentum balance equation (2.2). The antisymmetric part of the macrotensor $\langle\sigma_{ij}\rangle_j$ enters into the balance of the inner moment of momentum, as should have been expected.

3. A very important element in compiling the macroequations of motion is the calculation of the connection between the macrostresses $\langle\sigma_{ij}\rangle_j$ and the field of mean velocities $U_i(X_j, t)$. The instantaneous mean velocity in the Euler macrovolume V is introduced [8] as the mean mass and is referred to the center of mass with coordinates X_i

$$\langle\rho\rangle U_i = \langle\rho u_i\rangle \tag{3.1}$$

$$\langle\rho\rangle X_i = \langle\rho x_i\rangle, \qquad \langle\rho\xi_i\rangle = 0 \tag{3.2}$$

Then the field of local velocities $u_i(x_j, t)$ and the field of mean velocities $U_i(x_j, t)$ are represented in the volume as

$$u_i(x_j, t) = U_i(x_j, t) + v_i(x_j, t)$$
$$U_i(x_j, t) = U_i(X_j, t) + (\partial U_i / \partial X_j)(x_j - X_j)$$

where v_i is the pulsation velocity at the micropoint x_i.

According to (3.1), we have $\langle\rho v_i\rangle = 0$. Moreover, if we set $\langle v_i\rangle_j = 0$, i.e. accept the hypothesis of agreement between the results of volume and surface averaging for the vector quantities $\langle u_i\rangle_j = U_i$, then (2.3) becomes

$$\langle\sigma_{ij}\rangle = \frac{A_{ijkl}}{2}\left(\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k}\right) - \langle p\rangle\,\delta_{ij} \tag{3.3}$$

Therefore, the mean stress with respect to the volume is that part of the macrostress $\langle\sigma_{ij}\rangle_j$ which corresponds to the viscous stresses governed by a mean velocity field.

On the other hand, the local value of the stress is also representable as

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \frac{1}{2}A_{ijkl}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right) - (p - \langle p\rangle)\delta_{ij}$$

Hence, the deviator part of the macrostress $\langle\sigma_{ij}\rangle_j$ differs from the mean with respect to the volume because of the nonzero pulsation velocity gradients on the faces of the volume V. By virtue of (3.3), Eq. (2.7) also yields another representation for the macrostresses

$$\langle\sigma_{ij}\rangle_j - \langle\rho u_i u_j\rangle_j = \frac{A_{ijkl}}{2}\left(\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k}\right) - \langle p\rangle\,\delta_{ij} + \Sigma_{ij} \tag{3.4}$$

$$\Sigma_{ij} = -\langle\rho u_i u_j\rangle + \frac{\partial}{\partial t}\langle\rho u_i\xi_j\rangle + \frac{\partial}{\partial X_k}\langle\rho u_i u_k\xi_j\rangle_k - \frac{\partial}{\partial X_k}\langle\sigma_{ik}\xi_j\rangle_k - \langle F_i\xi_j\rangle$$

and the symmetric and antisymmetric parts can be separated in the additional tensor Σ_{ij}. If it is considered that the micromotion in the volume V is known, then the relationship (3.4) permits evaluation of the macrostress in terms of the surface quantities (integrals). It is easy to see that the representation (3.4) differs substantially from the computed Batchelor representation [2] (if the latter is used for the case of homogeneous fluid motion perturbed on a microlevel, which is under consideration).

4. Let us now mention the accepted interpretation of the macroquantities of the balance equations (2.1) – (2.5)

$$\langle \rho u_j \rangle_j = \langle \rho \rangle U_j,\ \langle \rho u_i u_j \rangle_j = -R_{ij} + \langle \rho \rangle U_i U_j,\ R_{ij} = -\langle \rho v_i v_j \rangle_j \neq -\langle \rho v_i v_j \rangle$$

$$\langle \varepsilon_{lik} \rho u_i \xi_k \rangle = M_l,\quad \langle \varepsilon_{lik}\, \rho u_i u_j \xi_k \rangle_j = -\mu_{lj} + M_l U_j$$

where R_{ij} are the Reynolds turbulent stresses, M_l is the internal angular momentum, μ_{lj} are the turbulent couple-stresses. As regards the quantities

$$T_{in} = \langle \varepsilon_{ijk}\ \sigma_{jn}\ \xi_k \rangle_n$$

they can be interpreted as viscous couple stresses due to the pulsations of viscous stresses on the appropriate faces of the volume V.

We also introduce the notation

$$\langle F_i \xi_k \rangle = \Phi_{ik},\quad \langle \sigma_{ij} \xi_k \rangle_j = \Pi_{ijk}$$

$$\langle \rho u_i \xi_k \rangle = \psi_{ik},\quad \langle \rho u_i u_j \xi_k \rangle_j = -\mu_{ijk} + \psi_{ik} U_j$$

where, in conformity with the Mindlin representations [10], Φ_{ik} is a mass couple force, ψ_{ik} is the intrinsic distortion, Π_{ijk} are viscous couple stresses, and μ_{ijk} are turbulent couple stresses.

Now, (3.4) becomes

$$\langle \sigma_{ij} \rangle_j + R_{ij} = \frac{1}{2} A_{ijkl} \left(\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k} \right) - \langle p \rangle\, \delta_{ij} -$$

$$\langle \rho v_i v_j \rangle + \frac{\partial \psi_{ij}}{\partial t} + U_l \frac{\partial \psi_{ij}}{\partial X_l} + \frac{\partial \mu_{ilj}}{\partial X_l} - \frac{\partial \Pi_{ilj}}{\partial X_l} - \Phi_{ij}$$

In the particular case of no inertial or mass forces, we obtain

$$\frac{1}{2} A_{ijkl} \left(\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k} \right) - \langle p \rangle\, \delta_{ij} = \frac{\partial \Pi_{ilj}}{\partial X_l} + \langle \sigma_{ij} \rangle_j$$

and this means that the representation of the mean stress with respect to the volume in terms of the surface integral [1] reduces to its being equal to the sum of the macrostresses and macrodivergences of the couple stress. Only in the absence of the couple stress (as well as the inertial and mass effects) will the means with respect to the volume and the surface stresses be equal. In the general case they are different and not only by the antisymmetric part.

The closing relations introduced here between the macrotensors and the corresponding kinematic macrovariables have the form of tensor relationships. Such quantities as the couple and double stresses are hence different from zero if there are additional (*) kinematic degrees of freedom (fields) in addition to the mean translational velocity U_i. The extraction of such quantities in the case of a turbulent fluid was given in [8 – 11]. The nature of the closing isotropic relations was developed in [12] for symmetric tensors and in [13] for asymmetric tensors.

If the hypothesis of equal mean stresses with respect to the volume and with respect to the surfaces is accepted, and thereby the condition of the disappearance of antisymmetric macrostress tensor components is accepted, then this will correspond to a more

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*) The additional dynamical and kinematical variables in generalized continuum mechanics are sometimes called microvariables (see [10]). It should be kept in mind that these microvariables (in contrast to those mentioned in this paper) are mean values.

particular type of closing relations (motions in the microscale).

The author is grateful to R.I. Nigmatulin for discussing the results.

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Translated by M.D.F.